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Generalized Lipschitz Classes and Asymptotic Behavior of Fourier Series

LING-YAU CHAN

Department of Industrial and Manufacturing Systems Engineering, University of Hong Kong, Hong Kong

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1. INTRODUCTION

It is well known that the L^p norm is an essential tool in the study of Fourier series. In letting $p \to \infty$ the L^p norm becomes the essential upper bound and L^p behavior formally becomes Lipschitz behavior. It is a natural way to generalize results on Fourier series by replacing the power-function by more general classes of functions. In this paper we shall focus on the application of a class of functions which has been used extensively in classical analysis by a number of authors [8, 9, 12, 13]. Following the notation in [6] we denote this class by $Y[a, b]$ ($a \leqslant b$), which is defined as the collection of all positive functions Φ defined on $(0, \infty)$ such that $\Phi(u)/u^a$ is nondecreasing and $\Phi(u)/u^b$ is nonincreasing. The class $Y[a, b]$ is known to be equivalent to other class of functions of the "power type" such as those introduced by Marcinkiewicz [11], Koizumi [10] , Woyczyński [15], and the class of $R-0$ varying functions [14]. For the details the readers are referred to [4, 6, 7]. The author [4] has proved the following integral relationship for functions in $Y[a, b]$:

THEOREM A. *Let $a \leqslant b$. Then a positive function Φ defined on $(0, \infty)$ belongs to the class $Y[a, b]$ if and only if $\Phi \in L(u_1, u_2)$ and $(a+1)\int_{u_1}^{u_2} \Phi(t)\,dt \leqslant u_2\Phi(u_2) - u_1\Phi(u_1) \leqslant (b+1)\int_{u_1}^{u_2} \Phi(t)\,dt$ for all u_1, $u_2 \in (0, \infty)$ and $u_1 < u_2$.*

From Theorem A we can deduce asumptotic relationships on partial sums (Section 3 below), and as a consequence the class $Y[a, b]$ is an appropriate choice for the purpose of establishing generalized Lipschitz behavior of Fourier series. In this paper, we shall generalize all the results in [5] by replacing the power-function by $\Phi \in Y[a, b]$.

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2. Generalized Lipschitz Classes

Definition. Let $\theta \in Y[\gamma_1, \gamma_2]$, where $0 \leqslant \gamma_1 \leqslant \gamma_2 < \infty$, and let $E \subset (-\infty, \infty)$. If G is a real-valued function defined on E, we shall define

(i) $G \in \operatorname{Lip} \theta$, if $\sup |G(x) - G(y)| = O(\theta(\delta))$ as $\delta \to 0^+$, and

(ii) $G \in \Lambda_*(\theta)$, if g is uniformly continuous in E and $\sup |G(x) + G(y) - 2G((x+y)/2)| = O(\theta(\delta))$ as $\delta \to 0^+$,

where the supremum in (i) and (ii) is taken for all $x, y \in E$ and $|x - y| \leqslant \delta$.

The classes $\operatorname{lip} \theta$ and $\lambda_*(\theta)$ are defined analogously when the above O's are replaced by o's. As a matter of fact, all the results in this paper stated with O's remain true when the O's are replaced by o's and vice versa.

Following the lines in [16, p. 44] it is not difficult to prove that:

Theorem B. *If $\theta \in Y[\gamma_1, \gamma_2]$ and $0 < \gamma_1 \leqslant \gamma_2 \leqslant 1$, then $\Lambda_*(\theta) \subset \operatorname{Lip} \theta_1$, where θ_1 is defined by $\theta_1(u) = \theta(u) |\log u|$. If in addition, $\gamma_2 < 1$, then $\Lambda_*(\theta) = \operatorname{Lip} \theta$.*

3. A Lemma on Partial Sums

In what follows we shall use the same letter K to denote various positive constants.

Lemma. *Let $c_k \geqslant 0$, $\Phi \in Y[\sigma_1, \sigma_2]$, $\psi \in Y[\delta_1, \delta_2]$.*

(i) *If $-\infty < \sigma_1 \leqslant \sigma_2 < \delta_1 \leqslant \delta_2 < \infty$, then*

$$W_n = \sum_{k=n}^{\infty} c_k = O(\Phi(1/n)) \qquad (n \to \infty) \tag{3.1}$$

implies

$$A_n = \sum_{k=1}^{n} \psi(k)\, c_k = O(\psi(n)\, \Phi(1/n)) \qquad (n \to \infty). \tag{3.2}$$

(ii) *If $0 < \sigma_1 \leqslant \sigma_2 < \infty$ and $-\infty < \delta_1 \leqslant \delta_2 < \infty$, then* (3.2) *implies* (3.1).

Proof of (i). Abel's transformation gives $A_n = \sum_{k=2}^{\infty} (\psi(k) - \psi(k-1))\, W_k + \psi(1)\, W_1 - \psi(n)\, W_{n+1}$. From $(k/(k-1))^{\delta_1} \psi(k-1) \leqslant \psi(k) \leqslant (k/(k-1))^{\delta_2}\, \psi(k-1)$ we have $-K\psi(k)/k \leqslant \psi(k) - \psi(k-1) \leqslant K\psi(k)/k$ $(k = 2, 3, \ldots)$.

Hence from (3.1) we have

$$\begin{aligned}
&\left|\sum_{k=2}^{n} (\psi(k)-\psi(k-1))\, W_k\right| \\
&\qquad \leqslant \sum_{k=2}^{n} K\psi(k)\, k^{-1}\Phi(1/k) \\
&\qquad \leqslant K \sum_{k=2}^{n} \int_{k-1}^{k} \psi(t)\, t^{-1}\Phi(1/t)\, dt \\
&\qquad\qquad \text{(since } \psi(u)\, u^{-1}\Phi(1/u)/u^{\delta_1 - 1 - \sigma_2} \text{ is nondecreasing)} \\
&\qquad \leqslant K(\delta_1 - \sigma_2)^{-1}(\psi(n)\, \Phi(1/n) - \psi(1)\, \Phi(1)) \qquad \text{(by Theorem A)} \\
&\qquad = O(\psi(n)\, \Phi(1/n)),
\end{aligned}$$

as $\delta_1 > \sigma_2$. This proves (i). The proof of (ii) is similar.

4. Asymptotic Properties of f and g

In what follows we shall let $a_n \geqslant 0$, $b_n \geqslant 0$ $(n = 1, 2, ...)$, $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx$, $g(x) = \sum_{n=1}^{\infty} b_n \sin nx$ be the Fourier series of f and g, respectively, and we let $\theta \in Y[\gamma_1, \gamma_2]$.

Theorem 1 (Cosine). *Let* $0 < \gamma_1 \leqslant \gamma_2 < \infty$. *Then*

$$f \in \operatorname{Lip} \theta \qquad (x \in [0, \pi]) \tag{4.1}$$

if and only if

$$\sum_{k=1}^{n} k^2 a_k = O(n^2 \theta(1/n))$$

and (4.2)

$$\sum_{k=1}^{n} k a_k \sin kx = O(n\theta(1/n)) \qquad \text{uniformly in } x, \text{ as } n \to \infty.$$

Proof. We have $\sin k(x+h) = \sin kx - (1-\cos kh) \sin kx + \cos kx \sin kh$, and when $|kh| \leqslant 1$ we have $1 - \cos kh = O(k^2h^2)$ and $\sin kh = kh + O(k^3h^3)$. Therefore, when $1 > h > 0$ we have

$$\begin{aligned}
&f(x+2h) - f(x) \\
&\qquad = \left(\sum_{k=1}^{1/h} + \sum_{k > 1/h}\right) 2a_k \sin k(x+h) \sin kh
\end{aligned}$$

$$= \sum_{k=1}^{1/h} 2kha_k \sin kx + O\left(\sum_{k=1}^{1/h} h^2k^2a_k\right) + O\left(\sum_{k=1}^{1/h} h^3k^3a_k\right)$$

$$+ O\left(\sum_{k=1/h}^{\infty} a_k\right)$$

$$= \sum_{k=1}^{1/h} 2kha_k \sin kx + h^2 O\left(\sum_{k=1}^{1/h} k^2a_k\right)$$

$$+ O\left(\sum_{k=1/h}^{\infty} a_k\right). \tag{4.3}$$

If (4.1) is satisfied, then $O(\theta(x)/x^2) = x^{-2}(f(0) - f(x))$

$$\geqslant \sum_{k=1}^{1/x} k^2a_k(1 - \cos kx)/(kx)^2$$

$$\geqslant (1 - \cos 1)/1^2 \sum_{k=1}^{1/x} k^2a_k,$$

where the last inequality is due to the fact $(1 - \cos t)/t^2$ decreases in $(0, 1]$. On putting $x = 1/n$ we obtain $\sum_{k=1}^{n} k^2a_k = O(n^2\theta(1/n))$. By Lemma (ii) we have $\sum_{k=n}^{\infty} a_k = O(\theta(1/n))$. Hence it follows from (4.3) that $\sum_{k=1}^{1/h} ka_k \sin kx$ is uniformly $O(h^{-1}\theta(h))$ as $h \to 0^+$.

Conversely, if (4.2) is satisfied, then by Lemma (ii) we have $\sum_{k=1/h}^{\infty} a_k = O(\theta(h))$, and (4.1) follows from (4.3). This proves Theorem 1.

If the condition on $\sum ka_k \sin kx$ is removed from (4.2), then (4.1) has to be replaced by $f \in \Lambda_*(\theta)$. To see this we only need to observe that for all $x, y \in [0, \pi]$, $x - y = 2h > 0$ we have $\cos kx + \cos ky - 2\cos k((x+y)/2) = -4\cos k(x-h)\sin^2(kh/2)$ and

$$|f(x) + f(y) - 2f((x+y)/2)|$$

$$\leqslant \left|\sum_{k=1}^{1/h} 4a_k \cos k(x-h)\sin^2(kh/2)\right| + 4\sum_{k=1/h}^{\infty} a_k$$

$$\leqslant \sum_{k=1}^{1/h} a_kk^2h^2 + 4\sum_{k=1/h}^{\infty} a_k \qquad (h \to 0+),$$

and that $f \in \Lambda_*(\theta)$ implies $f(0) - f(x) = O(\theta(x))$.

It follows from the Lemma that the condition $\sum_{k=1}^{n} k^2a_k = O(n^2\theta(1/n))$ is equivalent to $\sum_{k=n}^{\infty} a_k = O(\theta(1/n))$ when $\gamma_2 < 2$, which is in turn equivalent to $\sum_{k=1}^{n} ka_k = O(n\theta(1/n))$ when $\gamma_2 < 1$. Together with Theorem B we know that the above results cover [1, Theorems 3, 4] and the cosine series part of [1, Theorem 1; 4, Theorem 3].

THEOREM 2 (Sine). *Let* $0<\gamma_1\leqslant\gamma_2<\infty$. *Then*

$$g\in \operatorname{Lip}\theta \qquad (x\in(0,\pi]) \tag{4.4}$$

if and only if

$$\sum_{k=1}^{n} kb_k=O(n\theta(1/n)), \qquad as \quad n\to\infty. \tag{4.5}$$

Proof. Since a Fourier series can be term-wise integrated, if (4.4) is satisfied we have $\sum_{k=1}^{\infty} k^{-1}b_k(1-\cos kx)=\int_0^x g(t)\,dt=\int_0^x O(\theta(t))\,dt=O(x\theta(x))$ as $x\to 0^+$. Arguing as in the proof of [1, Theorem 1] we obtain (4.5). Conversely, if (4.5) is satisfied, by Lemma (ii) we have $\sum_{k=n}^{\infty} b_k=O(\theta(1/n))$ and when $0<h<1$ we have

$$\begin{aligned}|g(x+2h)-g(x)|&\leqslant 2\sum_{k=1}^{1/h} b_k\sin kh+2\sum_{k=1/h}^{\infty} b_k\\&\leqslant 2h\sum_{k=1}^{1/h} kb_k+2\sum_{k=1/h}^{\infty} b_k\\&=O(\theta(h)).\end{aligned}$$

This proves Theorem 2.

If $\gamma_2<1$, conditions (4.4), (4.5) are equivalent to $g\in\Lambda_*(\theta)$ and $\sum_{k=n}^{\infty} b_k=O(\theta(1/n))$. Hence Theorem 2 covers [1, Theorem 2] and the sine series part of [1, Theorem 1; 4, Theorem 3].

Applying the Lemma, Theorems 1 and 2, and mimicking the proofs in [5] we can generalize all the results proved in [5] by replacing the powers γ and α by $\theta\in Y[\gamma_1,\gamma_2]$. The following Theorem 3 which is a generalization of [5, Theorems 1 and 2] is quoted as an example, while all other details are omitted for brevity.

THEOREM 3. *Let* $0<\gamma_1\leqslant\gamma_2<\infty$. *If* $j\geqslant 1$ *is an integer, then*

$$\sum_{k=1}^{n} k^{2j+1}b_k=O(n^2\theta(1/n)) \tag{4.6}$$

if and only if

$$\begin{aligned}&g^{(2j-1)}(x) &&\textit{exists everywhere},\\&g^{(2m-2)}(0)=0 &&(2\leqslant m\leqslant j),\\&g^{(2j-1)}\in\Lambda_*(\theta).&&\end{aligned}\tag{4.7}$$

If in addition, $\gamma_2 < 1$, then the condition $g^{(2j-1)} \in \Lambda_*(\theta)$ in (4.7) is equivalent to $g^{(2j-1)} \in \operatorname{Lip} \theta$.

Let c be a real number, $x \in [0, \pi]$, and let u be either f or g, $s(kx) = \sin kx$ if $u = f$ and $s(kx) = \cos kx$ if $u = g$. Let $\sum_{k=n}^{\infty} \lambda_k = o(\theta(1/n))$, where $\lambda_k = a_k$ if $u = f$ and $\lambda_k = b_k$ if $u = g$. The following Theorem 4 is concerned with the derivative and symmetric derivative of u and it includes [1, Theorem 5] as a special case.

THEOREM 4. *Let $0 < \gamma_1 \leqslant \gamma_2 < 2$. Then $(u(x+h) - u(x))/h - c = o_x(\theta(|h|)/|h|)$ as $h \to 0$ if and only if $(u(x+h) - u(x-h))/(2h) - c = o_x(o_x(\theta(h)/h)$ as $h \to 0^+$, and if and only if $-\sum_{k=1}^{n} k\lambda_k s(kx) - c = o_x(n\theta(1/n))$ as $n \to \infty$.*

Theorem 4 can be readily proved by expanding $u(x+h) - u(x)$ and $u(x+h) - u(x-h)$ as in the proof of (4.3). As for the second symmetric derivative, it can be proved that $(u(x+h) + u(x-h) - 2u(x))/h^2 - c = o_x(\theta(h)/h^2)$ is equivalent to $-\sum_{k=1}^{n} k^2 \lambda_k c(kx) - c = o_x(n^2\theta(1/n))$ when $0 < \gamma_1 \leqslant \gamma_2 < 4$, where $c(kx) = \cos(kx)$ if $u = f$ and $c(kx) = \sin(kx)$ if $u = g$. For brevity the proof is omitted.

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